An Invariance Principle of G-Brownian Motion for the Law of the Iterated Logarithm under G-expectation

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Abstract

In this paper, we present the general invariance principle of G-Brownian motion for the law of the iterated logarithm under G-expectation, that is, for G-Brownian motion $(B_t)_{t\geq 0}$ in G-expectation space $(\Omega, L_G^1, \hat{\mathbb{E}})$, for any $n \geq 3$, let

$$\zeta_n(t) = (2n \log\log n)^{-\frac{1}{2}} B(nt), \quad \forall t \in [0,1],$$

$$K_\beta := \{x(\cdot) : x \in C([0,1]), x(0) = 0, \int_0^1 |\dot{x(t)}|^2 dt \leq \beta^2\}, \ \beta \in \mathbb{R}^+,$$

and if $\hat{\mathbb{E}}[B_1^2] = \bar{\sigma}^2, \ -\hat{\mathbb{E}}[-B_1^2] = \underline{\sigma}^2$, then
(I) the sequence $(\zeta_n)_{n\geq 3}$ is relatively norm-compact quasi-surely,
(II) $v\{C(\zeta_n) \subseteq K_{\bar{\sigma}}\} = 1$,
(III) $v\{C(\zeta_n) \supseteq K_{\underline{\sigma}}\} = 1$,
(IV) $\forall \beta \in [\underline{\sigma}, \bar{\sigma}], V\{C(\zeta_n) = K_\beta\} = 1$,
where $C(\zeta_n)$ denotes the cluster of sequence $(\zeta_n)_{n=3}^\infty$, (V, v) is the conjugate capacities generated by G-expectation $\hat{\mathbb{E}}[\cdot]$.

And we also give some examples as applications.

Keywords: Invariance Principle; Law of the Iterated Logarithm; Capacity; Sub-linear expectation; G-Brownian Motion

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1 Introduction

The classical law of the iterated logarithm (LIL for short) as fundamental limit theorems in probability theory play an important role in the development of probability theory and its applications. The original statement of the LIL obtained by Khinchine (1924) [7] is to a class of Bernoulli random variables. Kolmogorov (1929) [8] and Hartman-Wintner (1941) [5] extended Khinchine's result to large classes of independent random variables. Strassen (1964) [17] extended Hartman-Wintner's result to large classes of functional random variables, it is well known as the invariance principle for LIL which provide an extremely powerful tool in probability and statistical inference.

Starting with Strassen[17], a wide of literature has dealt with extension of the invariance priciple for the classical IID conditions are so strong that limit the applications of the invariance principle. So many papers have been to find weak dependence or nonstationary conditions which are not only enough to imply the invariance principle but also sufficiently general to be satisfied in typical applications, see for example [4],[9],[10],[11],[18].

On the other hand, the key in the proofs of the invariance principle is the additivity of the probabilities and the expectations. In practice, such additivity assumption is not feasible in many areas of applications because the uncertainty phenomena can not be modeled using additive probabilities or additive expectations. As an alternative to the traditional probability expectation, capacities or nonlinear probabilities expectations (for example Choquet integral, g-expectation) have been studied in many fields such as statistics, finance and economics.

Recently, motivated by the risk measures, super-hedge pricing and model uncertainty in finance, Peng [12]-[16] initiated the notion of independently and identically distributed (IID) random variables under sub-linear expectations. He also introduced the notion of G-normal distribution and G-Brownian motion as the counterpart of normal distribution and Brownian motion in linear case respectively. Under this framework, he proved one law of large numbers (LLN for short) and the central limit theorems (CLT for short) [15]. As well, Chen proved the strong LLN [1] and LIL [2] in this framework. G-expectation space is the most important sub-linear expectation space introduced by Peng [12], which take the role of Winener space in classical probability. Now more and more people are interested in G-expectation space or sub-linear expectation space.

A natural question is the following: Can the classical invariance principle for LIL be generalized under G-expectation space? The purpose of this paper is to investigate the invariance principle of G-Brownian motion for LIL adapting the Peng's IID notion under G-expectations space, of course we can not use the $\mathcal{X}^2$ distribution as in [17].

The remainder of this paper is organized as follows. In section 2, we recall some notions and properties in the G-expectation space, and prove some lemmas which will be useful in this paper. In section 3, we state and prove the main results of this paper, that is the invariance principle of G-Brownian motion for LIL under G-expectation. In section 4, we give some examples as applications of the new invariance principle.

2 Notations and Lemmas

In this section, we introduce some basic notations and lemmas. First, we shall recall briefly the notion of sub-linear expectation and IID random variables initiated by Peng in [15]. Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathcal{H}$ be a linear space of random variables on $(\Omega, \mathcal{F})$.

Definition 2.1 *[15] A function $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ is called a sub-linear expectation, if it satisfies the following four properties: for all $X, Y \in \mathcal{H}$*
(1) *Monotonicity:* $\mathbb{E}[X] \geq \mathbb{E}[Y]$ *if* $X \geq Y$;
(2) *Constant preserving:* $\mathbb{E}[c] = c,\ \forall c \in \mathbb{R}$;
(3) *Sub-additivity:* $\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$;
(4) *Positive homogeneity:* $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X],\ \forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sub-linear expectation space.

Given a sub-linear expectation $\mathbb{E}[\cdot]$, let us denote the conjugate expectation $\mathcal{E}[\cdot]$ of sub-linear expectation $\mathbb{E}[\cdot]$ by

$$\mathcal{E}[X] := -\mathbb{E}[-X], \qquad X \in \mathcal{H}.$$

Definition 2.2 *[15]* ***Independence and Identical distribution***
In a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random vector $Y \in \mathcal{H}^n$ is said to be independent from another random vector $X \in \mathcal{H}^m$ under $\hat{\mathbb{E}}[\cdot]$ if

$$\mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\mathbb{E}[\varphi(x,Y)]_{x=X}],\ \forall \varphi \in C_{l,lip}(\mathbb{R}^{m+n}).$$

Let X_1 and X_2 be two n-dimensional random variables in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$ respectively. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if

$$\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)]\ , \forall \varphi \in C_{l,lip}(\mathbb{R}^n).$$

If $\bar{X}$ is independent from X and $\bar{X} \overset{d}{=} X$, then $\bar{X}$ is said to be an independent copy of X.

Definition 2.3 *[15]* ***G-normal distributed***
A random variable X on a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called G-normal distributed, denoted by $X \sim \mathcal{N}(0, [\underline{\sigma}^2, \bar{\sigma}^2])$, if

$$aX + b\bar{X} \stackrel{d}{=} \sqrt{a^2 + b^2}X, \ \forall a, b \geq 0,$$

where $\bar{X}$ is an independent copy of X, $\bar{\sigma}^2 = \mathbb{E}[X^2]$ and $\underline{\sigma}^2 = \mathcal{E}[X^2]$. Here the letter G denotes the function $G(\alpha) := \frac{1}{2}\hat{\mathbb{E}}[\alpha X^2] = \frac{1}{2}(\bar{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-) : \mathbb{R} \to \mathbb{R}$.

Lemma 2.1 *[2] Suppose ξ is distributed to G normal $\mathcal{N}(0, [\underline{\sigma}^2, \bar{\sigma}^2])$, where $0 < \underline{\sigma} \leq \bar{\sigma} < \infty$. Let ϕ be a even and bounded continuous positive function, then for any $b \in \mathbb{R}$,*

$$e^{-\frac{b^2}{2\underline{\sigma}^2}}\mathcal{E}[\phi(\xi)] \leq \mathcal{E}[\phi(\xi - b)].$$

Definition 2.4 *[6]* ***G-Brownian motion***
Let $G(\cdot) : \mathbb{R} \to \mathbb{R}, G(\alpha) = \frac{1}{2}(\bar{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$, where $0 \leq \underline{\sigma} \leq \bar{\sigma} < \infty$. A stochastic process $(B_t)_{t\geq 0}$ in a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a G-Brownian motion if the following properties are satisfied:
(i) $B_0(\omega) = 0$;
(ii) For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is $\mathcal{N}(0, [s\underline{\sigma}^2, s\bar{\sigma}^2])$-distributed and is independent to $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t$.

• In the rest of this paper, we denote by $\Omega = C_0(\mathbb{R}^+)$ the space of all $\mathbb{R}$-valued continuous functions $(\omega_t)_{t\in\mathbb{R}^+}$ with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1].$$

• For every $\omega \in \Omega$, define the canonical process by $B_t(\omega) = \omega_t, \ t \geq 0$. The filtration generated by the canonical process $(B_t)_{t\geq 0}$ is defined by

$$\mathcal{F}_t = \sigma\{B_s, \ 0 \leq s \leq t\}, \ \mathcal{F} = \cup_{t\geq 0}\mathcal{F}_t.$$

• For each fixed $T \in [0, \infty)$, we set $\Omega_T := \{\omega_{\cdot\wedge T} : \omega \in \Omega\}$,

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1\wedge T}, \cdots, B_{t_n\wedge T}) : n \in \mathbb{N}, t_1, \cdots, t_n \in [0, \infty), \varphi \in C_{l,lip}(\mathbb{R}^n)\},$$

$$L_{ip}(\Omega) := \cup_{n=1}^{\infty} L_{ip}(\Omega_n).$$

• For any given monotonic increase and sub-linear function $G(\cdot) : \mathbb{R} \to \mathbb{R}$, there exist $\bar{\sigma} \geq \underline{\sigma} \geq 0$ such that $G(\alpha) = \frac{1}{2}(\bar{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$. We can construct (see [12],[13]) a consistent sublinear expectation called **G-expectation** $\hat{\mathbb{E}}[\cdot]$ on $L_{ip}(\Omega)$,

such that B_1 is G-normally distributed under $\hat{\mathbb{E}}[\cdot]$ and for each $s, t \geq 0$ and $t_1, \cdots, t_n \in [0, t]$ we have

$$\hat{\mathbb{E}}[\varphi(B_{t_1}, \cdots, B_{t_n}, B_{t+s} - B_t)] = \hat{\mathbb{E}}[\psi(B_{t_1}, \cdots, B_{t_n})],$$

where $\psi(x_1, \cdots, x_n) = \hat{\mathbb{E}}[\varphi(x_1, \cdots, x_n, \sqrt{s}B_1)]$. Under G-expectation $\hat{\mathbb{E}}[\cdot]$, the canonical process $(B_t)_{t\geq 0}$ is a G-Brownian motion.

We denote the completion of $L_{ip}(\Omega)$ under the norm $\|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{\frac{1}{p}}$ by $L_G^p(\Omega), p \geq 1$. And we also denote the extension by $\hat{\mathbb{E}}[\cdot]$. In the sequel, we consider the G-Brownian motion means the canonical process $(B_t)_{t\geq 0}$ under the **G-expectation space** $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$.

Let $\mathcal{A}_{0,\infty}$ denote the set of all $[\underline{\sigma}, \bar{\sigma}]$-valued, $\mathcal{F}_t$-adapted processes on the interval $[0, 1]$. For each fixed $\theta \in \mathcal{A}_{0,\infty}$, set P_θ, the law of the process $(\int_0^t \theta_s dB_s)_{t\geq 0}$ under the Wiener measure P. We denote by

$$\mathcal{P} = \{P_\theta : \theta \in \mathcal{A}_{0,\infty}\},$$

and define

$$V(A) := \sup_{\theta\in\mathcal{A}_{0,\infty}} P_\theta(A), \ v(A) := \inf_{\theta\in\mathcal{A}_{0,\infty}} P_\theta(A), \ \forall A \in \mathcal{B}(\Omega).$$

It is easy to check that

$$V(A) + v(A^c) = 1, \ \forall A \in \mathcal{B}(\Omega),$$

where A^c is the complement set of A. Through this paper, we assume that (V, v) is the conjugate capacities generated by G-expectation $\hat{\mathbb{E}}[\cdot]$. From Denis et al.[3], we know that $\mathcal{P}$ is tight. For each $X \in L^0(\Omega)$ (the space of all Borel measurable real functions on Ω) such that $E_\theta(X)$ exists for any $\theta \in \mathcal{A}_{0,\infty}$, define the upper expectation

$$\bar{\mathbb{E}}[X] := \sup_{\theta\in\mathcal{A}_{0,\infty}} E_\theta(X).$$

From Denis et al. [3], for all $X \in L_G^1(\Omega)$, it holds that $\bar{\mathbb{E}}[X] = \hat{\mathbb{E}}[X]$.

Definition 2.5 *[3]* ***quasi-surely***
A set D is polar set if $V(D) = 0$ and a property holds "quasi-surely" (q.s. for short) if it holds outside a polar set.

Lemma 2.2 *[2]* ***Borel-Cantelli lemma***
Let $\{A_n, n \geq 1\}$ be a sequence of events in $\mathcal{F}$ and (V, v) be a pair of capacities generated by G-expectation $\hat{\mathbb{E}}[\cdot]$.

(1) *If* $\sum_{n=1}^{\infty} V(A_n) < \infty$, *then* $V(\cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i) = 0$.
(2) *Suppose that* $\{A_n, n \geq 1\}$ *are pairwise independent with respect to* V, *that is*

$$V(\cap_{n=1}^{\infty} A_n^c) = \prod_{n=1}^{\infty} V(A_n^c).$$

If $\sum_{n=1}^{\infty} v(A_n) = \infty$, *then* $v(\cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i) = 1$.

Lemma 2.3 (1) *Let* X_1 *and* X_2 *be two real random variables in G-expectation spaces* $(\Omega, L_G^1, \hat{\mathbb{E}})$. *If* $X_1 \overset{d}{=} X_2$, *then*

$$\bar{\mathbb{E}}[\varphi(X_1)] = \bar{\mathbb{E}}[\varphi(X_2)], \text{ i.e. } V(X_1 \in A) = V(X_2 \in A),$$

where $\varphi(x) = I_A(x)$, A *is an interval of the following types:* $(-\infty, a)$, $(-\infty, a]$, (a, b), $(a, b]$, $[a, b)$, $[a, b]$, $[b, \infty)$, *or* (b, ∞) *with* $(a, b) \in \mathbb{R}^2$ *and* $a < b$.

(2) *In G-expectation space* $(\Omega, L_G^1, \hat{\mathbb{E}})$, *a real random variable* Y *is independent from another real random variable* X *under* $\hat{\mathbb{E}}[\cdot]$, *then*

$$\bar{\mathbb{E}}[\varphi(X, Y)] = \bar{\mathbb{E}}[\bar{\mathbb{E}}[\varphi(x, Y)]_{x=X}], \text{ i.e. } V(X \in A, Y \in B) = V(X \in A)V(Y \in B),$$

where $\varphi(x, y) = I_A(x)I_B(y)$, A *is an interval of the following types:* $(-\infty, a)$, $(-\infty, a]$, (a, b), $(a, b]$, $[a, b)$, $[a, b]$, $[b, \infty)$, *or* (b, ∞); B *is an interval of the following types:* $(-\infty, c)$, $(-\infty, c]$, (c, d), $(c, d]$, $[c, d)$, $[c, d]$, $[d, \infty)$ *or* (d, ∞) *with* $(a, b, c, d) \in \mathbb{R}^4$ *and* $a < b, c < d$.

Proof. (1). We just consider the case $a = y$, $A = (-\infty, y]$, other cases can be proved similarly. Let $\mathbb{F}(y) := \bar{\mathbb{E}}[I_{X_1 \leq y}] = V(X_1 \leq y)$, then $\mathbb{F}$ is a continuous function on $\mathbb{R}$ from lemma 8 in page 143 of [3]. So for each $y \in \mathbb{R}$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\mathbb{F}(\bar{y}) - \mathbb{F}(y)| < \varepsilon, \ \forall \bar{y} \in [y - \delta, y + \delta].$$

Now we define two auxiliary functions

$$f(x) = \begin{cases} 1 & x \in (-\infty, y - \delta); \\ \frac{(y-x)}{\delta} & x \in [y - \delta, y]; \\ 0 & x \in (y, \infty), \end{cases} \quad g(x) = \begin{cases} 1 & x \in (-\infty, y); \\ \frac{(y+\delta-x)}{\delta} & x \in [y, y + \delta]; \\ 0 & x \in (y + \delta, \infty). \end{cases}$$

Because $X_1 \overset{d}{=} X_2$, using the monotonicity of $\bar{\mathbb{E}}[\cdot]$, we get

$$\mathbb{F}(y - \delta) \leq \hat{\mathbb{E}}[f(X_1)] = \hat{\mathbb{E}}[f(X_2)] \leq \bar{\mathbb{E}}[I_{X_2 \leq y}] \leq \hat{\mathbb{E}}[g(X_2)] = \hat{\mathbb{E}}[g(X_1)] \leq \mathbb{F}(y + \delta).$$

Therefore

$$-\varepsilon \leq \mathbb{F}(y-\delta) - \mathbb{F}(y) \leq \bar{\mathbb{E}}[I_{X_2 \leq y}] - \bar{\mathbb{E}}[I_{X_1 \leq y}] \leq \mathbb{F}(y+\delta) - \mathbb{F}(y) \leq \varepsilon.$$

By the arbitrariness of ε we get $\bar{\mathbb{E}}[I_{X_2 \leq y}] = \bar{\mathbb{E}}[I_{X_1 \leq y}]$, that is $V(X_1 \leq y) = V(X_2 \leq y)$.

(2). We only consider the case $A = (-\infty, a],\ B = (-\infty, c]$, other cases can be proved in the same way. For $\mathcal{P}$ being compact, we get $\mathbb{F}(a,c) := \bar{\mathbb{E}}[I_{X \in A} \cdot I_{Y \in B}] = V(X \leq a, Y \leq c)$ is a continuous function on $\mathbb{R}^2$ also from lemma 8 in page 143 of [3]. So for each $(a,c) \in \mathbb{R}^2$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\mathbb{F}(\bar{a}, \bar{c}) - \mathbb{F}(a,c)| < \varepsilon,\ \forall (\bar{a}, \bar{c}) \in [a-\delta, a+\delta] \times [c-\delta, c+\delta].$$

Similarly as (1) we define two auxiliary functions

$$f(x,y) = \begin{cases} 1 & (x,y) \in (-\infty, a-\delta) \times (-\infty, c-\delta); \\ 0 & (x,y) \in (a, \infty) \times (c, \infty); \\ \frac{(a-x)(c-y)}{\delta^2} & \text{others,} \end{cases}$$

$$g(x,y) = \begin{cases} 1 & (x,y) \in (-\infty, a) \times (-\infty, c); \\ 0 & (x,y) \in (a+\delta, \infty) \times (c+\delta, \infty); \\ \frac{(a+\delta-x)(c+\delta-y)}{\delta^2} & \text{others.} \end{cases}$$

Obviously,

$$f(x,y) \leq \varphi(x,y) = I_{X \leq a} \cdot I_{Y \leq c} \leq g(x,y.)$$

For Y is independent from X under $\hat{\mathbb{E}}[\cdot]$, we get

$$\mathbb{F}(a-\delta, c-\delta) \leq \hat{\mathbb{E}}[f(X,Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[f(x,Y)]_{x=X}] \leq \bar{\mathbb{E}}[\bar{\mathbb{E}}[\varphi(x,Y)]_{x=X}],$$

while

$$\bar{\mathbb{E}}[\bar{\mathbb{E}}[\varphi(x,Y)]_{x=X}] \leq \hat{\mathbb{E}}[\hat{\mathbb{E}}[g(x,Y)]_{x=X}] = \hat{\mathbb{E}}[g(X,Y)] \leq \mathbb{F}(a+\delta, c+\delta).$$

It deduce that

$$-\varepsilon \leq \mathbb{F}(a-\delta, c-\delta) - \mathbb{F}(a,c) \leq \bar{\mathbb{E}}[\bar{\mathbb{E}}[\varphi(x,Y)]_{x=X}] - \bar{\mathbb{E}}[\varphi(X,Y)] \leq \mathbb{F}(a+\delta, c+\delta) - \mathbb{F}(a,c) \leq \varepsilon.$$

By the arbitrariness of ε we get $\bar{\mathbb{E}}[\varphi(X,Y)] = \bar{\mathbb{E}}[\bar{\mathbb{E}}[\varphi(x,Y)]_{x=X}]$, that is $V(X \leq a, Y \leq c) = V(X \leq a)V(Y \leq c)$. $\Box$

Remark: A similar result as (1) of Lemma 2.3 can be proved similarly when X_1, X_2 are in two sub-linear expectation spaces respectively and one of the probability set generated by the sub-linear expectation is weakly compact. Meanwhile, the conclusion (2) of Lemma 2.3, of course, holds true if X, Y are in a sub-linear expectation spaces and the probability set generated by the sub-linear expectation is weakly compact, the proof is also similarly.

Lemma 2.4 *For any $s \le t$, we have for almost surely $y \in \mathbb{R}^+$*

$$V(|B_s| \ge y) \le V(|B_t| \ge y).$$

Proof. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W_t)_{t\ge 0}$ be Brownian motion in this space, $(\mathcal{F}_t)_{t\ge 0}$ is the filtration generated by $(W_t)_{t\ge 0}$. Denis et al. [3] have proved

$$\hat{\mathbb{E}}[\varphi(B_1)] = \sup_{\theta\in\Theta} E_P[\varphi(\int_0^1 \theta_s dW_s)],\ \forall \varphi \in C_{b,lip}(\mathbb{R}), \tag{1}$$

where Θ denote the collection of all $[\underline{\sigma}, \bar{\sigma}]$-valued $\mathcal{F}_t$-adapted process on interval [0,1]. To proof the lemma, we fist prove

$$\bar{\mathbb{E}}[I_{B_1 \le y}] = \sup_{\theta\in\Theta} E_P[I_{\int_0^1 \theta_s dW_s \le y}]. \tag{2}$$

Define $\mathbb{F}(y) := \sup_{\theta\in\Theta} E_P[I_{\int_0^1 \theta_s dW_s \le y}]$, then $\mathbb{F}(\cdot)$ is a not decreasing function on $\mathbb{R}$. So $\mathbb{F}(\cdot)$ is almost surely continuous. Take y is the continuous point of $\mathbb{F}(\cdot)$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\mathbb{F}(\bar{y}) - \mathbb{F}(y)| < \varepsilon,\ \forall \bar{y} \in [y-\delta, y+\delta].$$

Using the same auxiliary functions in the proof of the first part of Lemma (2.3) and equality (1), we get

$$\mathbb{F}(y-\delta) \le \hat{\mathbb{E}}[f(B_1)] \le \bar{\mathbb{E}}[I_{B_1 \le y}] \le \hat{\mathbb{E}}[g(B_1)] \le \mathbb{F}(y+\delta).$$

Therefore

$$-\varepsilon \le \mathbb{F}(y-\delta) - \mathbb{F}(y) \le \bar{\mathbb{E}}[I_{B_1 \le y}] - \mathbb{F}(y) \le \mathbb{F}(y+\delta) - \mathbb{F}(y) \le \varepsilon.$$

By the arbitrariness of ε we get $\bar{\mathbb{E}}[I_{B_1 \le y}] = \mathbb{F}(y)$, that is equality (2) hold.

Similarly, we can show for almost surely $y \in \mathbb{R}^+$ the following equality (3) hold

$$\bar{\mathbb{E}}[I_{|B_1| \ge y}] = \sup_{\theta\in\Theta} E_P[I_{|\int_0^1 \theta_s dW_s| \ge y}]. \tag{3}$$

Since $B_t \overset{d}{=} \sqrt{t} B_1$, from Lemma 2.3 and equality (3) we have

$$V(|B_t| \ge y) = \bar{\mathbb{E}}[I_{|B_t| \ge y}] = \bar{\mathbb{E}}[I_{|B_1| \ge \frac{y}{\sqrt{t}}}] = \sup_{\theta\in\Theta} E_P[I_{|\int_0^1 \theta_s dW_s| \ge \frac{y}{\sqrt{t}}}].$$

Hence, for any $s \le t$, we have for almost surely $y \in \mathbb{R}^+$, $V(|B_s| \ge y) \le V(|B_t| \ge y)$. $\square$

Lemma 2.5 *If $\underline{\sigma} > 0$, then the paths of G-Brownian motion are q.s. nowhere differentiable.*

Proof. Suppose $B.(\omega)$ is differentiable at s, then there exists $\delta > 0, l \geq 1$, such that $|B_t(\omega) - B_s(\omega)| < l|t-s|$ for any $|t-s| < \delta$. From the definition of quadratic variation process of G-Brownian motion [12], it follows that

$$\langle B\rangle_{s+\delta}(\omega) - \langle B\rangle_s(\omega) = \lim_{\mu(\pi_t^n)\to 0} \sum_{i=0}^{n} |B_{t_{i+1}}(\omega) - B_{t_i}(\omega)|^2 \leq \lim_{\mu(\pi_t^n)\to 0} l^2 \sum_{i=0}^{n} |t_{i+1} - t_i|^2 = 0.$$

Meanwhile Peng in [16] has show that $\langle B\rangle_{s+\delta}(\omega) - \langle B\rangle_s > \underline{\sigma}\delta > 0$ q.s.. We deduce therefore that $V\{\omega : B.(\omega)$ is differentiable at $s\} = 0$. So the lemma holds. $\square$

3 The invariance principle of G-Brownian motion

In this section we will consider the invariance principle of G-Brownian motion for the LIL under G-expectation. First let us give some basic notations. Let $B(t)_{t\geq 0}$ be the G-Brownian motion, $B_t \sim \mathcal{N}(0, [t\underline{\sigma}^2, t\bar{\sigma}^2])$. Define

$$\zeta_n(t) = (2n \log\log n)^{-\frac{1}{2}} B(nt), \quad \forall t \in [0,1], n \geq 3.$$

Let $C([0,1])$ be the banach space of continuous maps from $[0,1]$ to $\mathbb{R}$ endowed with the supremum norm $\|\cdot\|$, using the enuclidean norm in $\mathbb{R}$. ζ_n is then a random variable with values in $C([0,1])$. For any $\beta \in \mathbb{R}^+$, define

$$K_\beta := \{x(\cdot) : x \in C([0,1]), x(0) = 0, \int_0^1 |x(\dot{t})|^2 dt \leq \beta^2\}.$$

Theorem 3.1 *Let $C(\zeta_n)$ denotes the cluster of sequence $(\zeta_n)_{n=3}^\infty$, then*
(I) The sequence $(\zeta_n)_{n\geq 3}$ is relatively norm-compact q.s..
(II) $v\{C(\zeta_n) \subseteq K_{\bar{\sigma}}\} = 1$.
(III) $v\{C(\zeta_n) \supseteq K_{\underline{\sigma}}\} = 1$.
(IV) $\forall \beta \in [\underline{\sigma}, \bar{\sigma}], V\{C(\zeta_n) = K_\beta\} = 1$.

Proof. (I) and (II). For any $\varepsilon > 0$, let

$$K_{\bar{\sigma}}^\varepsilon := \{x(\cdot) : \ x \in C([0,1]), d(x, K_{\bar{\sigma}}) \leq \varepsilon\}.$$

Moreover, let η_n be the random variable in $C([0,1])$ obtained by interpolating the points $\zeta_n(\frac{i}{m})$ at $\frac{i}{m}$ $(i = 1, \cdots, m)$, where m is a positive integer which will be decided in the later. For any $\varepsilon_1 > 0$, we have

$$\begin{aligned}
&V\{\zeta_n \notin K_{\bar{\sigma}}^\varepsilon\} \\
&\leq V\{\eta_n \notin K_{\bar{\sigma}+\varepsilon_1}\} + V\{\eta_n \in K_{\bar{\sigma}+\varepsilon_1}, \ \zeta_n \notin K_{\bar{\sigma}}^\varepsilon\} \\
&= I_1 + I_2.
\end{aligned}$$

For any $\lambda > 0$, by Chebyshev's inequality and Lemma 2.3, we get

$$
\begin{aligned}
I_1 &= V\{\int_0^1 |\dot{\eta_n(t)}|^2 dt > (\bar{\sigma}+\varepsilon_1)^2\} \\
&= V\left\{\frac{\sum_{i=1}^m [B(\frac{ni}{m}) - B(\frac{n(i-1)}{m})]^2}{2n\log\log n/m} > (\bar{\sigma}+\varepsilon_1)^2\right\} \\
&= V\left\{\frac{\sum_{i=1}^m [B(\frac{ni}{m}) - B(\frac{n(i-1)}{m})]^2}{n\bar{\sigma}^2/m} > 2(1+\frac{\varepsilon_1}{\bar{\sigma}})^2 \log\log n\right\} \\
&\le \exp(-2\lambda(1+\frac{\varepsilon_1}{\bar{\sigma}})^2\log\log n)\bar{\mathbb{E}}\left[\exp\left(\frac{\sum_{i=1}^m \lambda[B(\frac{ni}{m}) - B(\frac{n(i-1)}{m})]^2}{n\bar{\sigma}^2/m}\right)\right] \\
&= \exp(-2\lambda(1+\frac{\varepsilon_1}{\bar{\sigma}})^2\log\log n)\prod_{i=1}^m \bar{\mathbb{E}}\left[\exp\left(\frac{\lambda[B(\frac{ni}{m}) - B(\frac{n(i-1)}{m})]^2}{n\bar{\sigma}^2/m}\right)\right] \\
&= \exp(-2\lambda(1+\frac{\varepsilon_1}{\bar{\sigma}})^2\log\log n)[\bar{\mathbb{E}}\exp(\frac{\lambda B(\frac{n}{m})^2}{n\bar{\sigma}^2/m})]^m
\end{aligned}
$$

For each $\varepsilon_1 > 0$, we choose $\lambda(\varepsilon_1) \in (0, \frac{1}{2})$ such that $\beta_1 := 2\lambda(1+\frac{\varepsilon_1}{\bar{\sigma}})^2 > 1$, therefore

$$
C(\varepsilon_1) := \bar{\mathbb{E}}\exp(\frac{\lambda B(\frac{n}{m})^2}{n\bar{\sigma}^2/m}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(\lambda y^2)\exp(-y^2/2)dy < \infty.
$$

Hence

$$
I_1 \le C(\varepsilon_1)^m \exp(-\beta_1 \log\log n). \tag{4}
$$

Meanwhile,

$$
\begin{aligned}
I_2 &= V\{\eta_n \in K_{\bar{\sigma}+\varepsilon_1},\ \zeta_n \notin K_{\bar{\sigma}}^{\varepsilon}\} \\
&\le V\{\eta_n \in K_{\bar{\sigma}+\varepsilon_1},\ \|\zeta_n - \frac{\bar{\sigma}}{\bar{\sigma}+\varepsilon_1}\eta_n\| \ge \varepsilon\} \\
&= \sup_{P\in\mathcal{P}} P\{\eta_n \in K_{\bar{\sigma}+\varepsilon_1},\ \|\zeta_n - \frac{\bar{\sigma}}{\bar{\sigma}+\varepsilon_1}\eta_n\| \ge \varepsilon\}
\end{aligned}
$$

Define the random variable T by

$$
T := \begin{cases} \min\{t:\ t\in[0,1], |\zeta_n(t) - \frac{\bar{\sigma}}{\bar{\sigma}+\varepsilon_1}\eta_n(t)| \ge \varepsilon\}, & \text{if this set is nonempty;} \\ 2, & \text{otherwise,} \end{cases}
$$

and let F_P be its distribution function under P, thus

$$
\begin{aligned}
I_2 &\le \sup_{P\in\mathcal{P}} \int_0^1 P\{\eta_n \in K_{\bar{\sigma}+\varepsilon_1} | T = t\} dF_P(t) \\
&= \sup_{P\in\mathcal{P}} \int_0^1 P\{\eta_n \in K_{\bar{\sigma}+\varepsilon_1},\ |\zeta_n(t) - \frac{\bar{\sigma}}{\bar{\sigma}+\varepsilon_1}\eta_n(t)| = \varepsilon | T = t\} dF_P(t).
\end{aligned}
$$

Let $i(t)$ denote the smallest integer i with $i/m \geq t$, the statement $\eta_n \in K_{\bar{\sigma}+\varepsilon_1}$ implies

$$|\eta_n(\frac{i(t)}{m}) - \eta_n(t)| \leq \frac{\bar{\sigma}+\varepsilon_1}{\sqrt{m}}.$$

Together with

$$|\zeta_n(t) - \frac{\bar{\sigma}}{\bar{\sigma}+\varepsilon_1}\eta_n(t)| = \varepsilon,$$

we have

$$\begin{aligned}
&|\zeta_n(\frac{i(t)}{m}) - \zeta_n(t)| \\
&\geq |\eta_n(t) - \zeta_n(t)| - |\eta_n(\frac{i(t)}{m}) - \eta_n(t)| \\
&\geq \frac{\bar{\sigma}+\varepsilon_1}{\bar{\sigma}}|\frac{\bar{\sigma}}{\bar{\sigma}+\varepsilon_1}\eta_n(t) - \zeta_n(t) - \frac{-\varepsilon_1}{\bar{\sigma}+\varepsilon_1}\zeta_n(t)| - \frac{\bar{\sigma}+\varepsilon_1}{\sqrt{m}} \\
&\geq (1+\frac{\varepsilon_1}{\bar{\sigma}})\varepsilon - \frac{\varepsilon_1|B(nt)|}{\bar{\sigma}\sqrt{2n\log\log n}} - \frac{\bar{\sigma}+\varepsilon_1}{\sqrt{m}} \\
&\geq \varepsilon/2 \qquad q.s.
\end{aligned}$$

where the last inequality is obtained by LIL of chen [2] and choose ε_1 close to 0 and m be sufficiently large. For $\lambda > 0$, using lemma 2.4 it follows that

$$\begin{aligned}
&I_2 \leq \sup_{P\in\mathcal{P}} \int_0^1 P\{|\zeta_n(\frac{i(t)}{m}) - \zeta_n(t)| \geq \varepsilon/2 | T = t\} dF_P(t) \\
&\leq \sup_{P\in\mathcal{P}} V\{|\zeta_n(\frac{1}{m})| \geq \varepsilon/2\} \int_0^1 dF_P(t) \\
&\leq V\left\{\frac{|B(\frac{n}{m})|}{\sqrt{\frac{n\bar{\sigma}^2}{m}}} \geq \frac{\varepsilon\sqrt{2m\log\log n}}{2\bar{\sigma}}\right\} \\
&\leq \exp(-\frac{\lambda\varepsilon^2 m\log\log n}{2\bar{\sigma}^2})\bar{\mathbb{E}}[\exp(\frac{\lambda B(\frac{n}{m})^2}{\frac{n\bar{\sigma}^2}{m}})].
\end{aligned}$$

Choosing $\lambda(\varepsilon) \in (0, 1/2)$ and m such that $\beta_2 := \lambda\varepsilon^2 m/\bar{\sigma}^2 > 1$, as before there exist $D(\varepsilon) > 0$ such that

$$I_2 \leq D(\varepsilon)\exp(-\beta_2 \log\log n). \tag{5}$$

From inequalities (4) and (5)we have $\beta := \beta_1 \wedge \beta_2 > 1$ and

$$V\{\zeta_n \notin K_{\bar{\sigma}}^{\varepsilon}\} \leq (C(\varepsilon_1)^m + D(\varepsilon))\exp(-\beta\log\log n).$$

If $n_k = \lfloor c^k \rfloor + 1$, where $c > 1$, $\lfloor c \rfloor$ is the largest integer not greater than c, then

$$\sum_{k=1}^{\infty} V\{\zeta_{n_k} \notin K_{\bar{\sigma}}^{\varepsilon}\} \leq (C(\varepsilon_1)^m + D(\varepsilon))(\log c)^{-\beta} \sum_{k=1}^{\infty} k^{-\beta} < \infty.$$

By Borel-Cantelli lemma, we have

$$V\{\bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} (\zeta_{n_k} \notin K_{\bar{\sigma}}^{\varepsilon})\} = 0,$$

in other words,

$$v\{\bigcup_{i=1}^{\infty} \bigcap_{k=i}^{\infty} (\zeta_{n_k} \in K_{\bar{\sigma}}^{\varepsilon})\} = 1.$$

For c sufficient close to 1 this implies that

$$v\{\bigcup_{i=1}^{\infty} \bigcap_{k=i}^{\infty} (\zeta_n \in K_{\bar{\sigma}}^{2\varepsilon})\} = 1.$$

Hence, $v(C(\zeta_n) \subseteq K_{\bar{\sigma}}) = 1$ and for any $\varepsilon > 0$, $(\zeta_n)_{n \geq 3}$ exists a relatively compact 2ε-net q.s.. This shows that $(\zeta_n)_{n \geq 3}$ is relatively compact q.s.. The proof of (I) and (II) is complete.

(III). For any $0 < \beta < \underline{\sigma}$, any given $x \in K_\beta$, let $m \geq 1$ be an integer, $\varepsilon_0 > 0$ and $A_n = \cap_{i=1}^m A_n^i$, where

$$A_n^i = \left\{ \left| \zeta_n(\frac{i}{m}) - \zeta_n(\frac{i-1}{m}) - \left(x(\frac{i}{m}) - x(\frac{i-1}{m}) \right) \right| < \varepsilon_0 \right\}, \quad i = 1, \cdots, m.$$

By the stationary increments property of G-Brownian motion and Lemma 2.3, we have

$$\begin{aligned} v(A_n^i) &= v\left\{ \left| \frac{B(\frac{n}{m})}{\sqrt{2n \log\log n}} - \left(x(\frac{i}{m}) - x(\frac{i-1}{m}) \right) \right| \leq \varepsilon_0 \right\} \\ &= v\left\{ \left| \frac{\bar{B}(n)}{\sqrt{2n \log\log n}} - \left(x(\frac{i}{m}) - x(\frac{i-1}{m}) \right) \right| \leq \varepsilon_0 \right\}, \end{aligned}$$

where $\bar{B}(t) = B(\frac{t}{m}) \sim \mathcal{N}(0, [\frac{\underline{\sigma}^2 t}{m}, \frac{\bar{\sigma}^2 t}{m}])$. Let us choose $n_k = k^{k^\alpha}$ for $k \geq 1$ where $0 < \alpha < \frac{1}{2m}$. Then we obtain that

$$\begin{aligned} v(A_{n_k}^i) \geq\ & v\left\{ \left| \frac{\bar{B}(n_k) - \bar{B}(n_{k-1})}{\sqrt{2n_k \log\log n_k}} - \left(x(\frac{i}{m}) - x(\frac{i-1}{m}) \right) \right| \leq \frac{\varepsilon_0}{2} \right\} \\ & \cdot v\left\{ \frac{|\bar{B}(n_{k-1})|}{\sqrt{2n_{k-1} \log\log n_{k-1}}} \frac{\sqrt{2n_{k-1} \log\log n_{k-1}}}{\sqrt{2n_k \log\log n_k}} \leq \frac{\varepsilon_0}{2} \right\} \end{aligned}$$

For each $t > 0$, let

$$N_k := \lfloor (n_{k+1} - n_k)^2 t^2 / (2n_{k+1} \log\log n_{k+1}) \rfloor,$$

$$l_k := \lfloor 2t^{-2} n_{k+1} \log\log n_{k+1} / (n_{k+1} - n_k) \rfloor,$$

$$r_k := \sqrt{2n_{k+1} \log\log n_{k+1}} / (t l_k).$$

We have

$$\begin{aligned}
& v\left\{ \left| \frac{\bar{B}(n_k) - \bar{B}(n_{k-1})}{\sqrt{2n_k \log\log n_k}} - \left(x(\frac{i}{m}) - x(\frac{i-1}{m}) \right) \right| \leq \frac{\varepsilon_0}{2} \right\} \\
= \; & v\left\{ x(\frac{i}{m}) - x(\frac{i-1}{m}) - \frac{\varepsilon_0}{2} \leq \frac{\bar{B}(n_k - n_{k-1})}{\sqrt{2n_k \log\log n_k}} \leq x(\frac{i}{m}) - x(\frac{i-1}{m}) + \frac{\varepsilon_0}{2} \right\} \\
\geq \; & v\left\{ x(\frac{i}{m}) - x(\frac{i-1}{m}) - \frac{\varepsilon_0}{4} \leq \frac{\bar{B}(N_{k-1} l_{k-1})}{\sqrt{2n_k \log\log n_k}} \leq x(\frac{i}{m}) - x(\frac{i-1}{m}) + \frac{\varepsilon_0}{4} \right\} \\
\cdot \; & v\left\{ -\frac{\varepsilon_0}{4} \leq \frac{\bar{B}(n_k - n_{k-1}) - \bar{B}(N_{k-1} l_{k-1})}{\sqrt{2n_k \log\log n_k}} \leq \frac{\varepsilon_0}{4} \right\} \\
\geq \; & v\left\{ t l_{k-1} \left(x(\frac{i}{m}) - x(\frac{i-1}{m}) - \frac{\varepsilon_0}{4} \right) \leq \frac{\bar{B}(N_{k-1} l_{k-1})}{r_{k-1}} \leq t l_{k-1} \left(x(\frac{i}{m}) - x(\frac{i-1}{m}) + \frac{\varepsilon_0}{4} \right) \right\} \\
\cdot \; & v\left\{ -\frac{\varepsilon_0}{4} \leq \frac{\bar{B}(n_k - n_{k-1} - N_{k-1} l_{k-1})}{\sqrt{2n_k \log\log n_k}} \leq \frac{\varepsilon_0}{4} \right\} \\
\geq \; & v\left\{ \left(x(\frac{i}{m}) - x(\frac{i-1}{m}) \right) t - \frac{\varepsilon_0 t}{4} \leq \frac{\bar{B}(N_{k-1})}{r_{k-1}} \leq \left(x(\frac{i}{m}) - x(\frac{i-1}{m}) \right) t + \frac{\varepsilon_0 t}{4} \right\}^{l_{k-1}} \\
\cdot \; & v\left\{ -\frac{\varepsilon_0}{4} \leq \frac{\bar{B}(n_k - n_{k-1} - N_{k-1} l_{k-1})}{\sqrt{2n_k \log\log n_k}} \leq \frac{\varepsilon_0}{4} \right\} \\
\geq \; & \left(\mathcal{E}\left[\phi\left(\frac{\bar{B}(N_{k-1})}{r_{k-1}} - (x(\frac{i}{m}) - x(\frac{i-1}{m}))t \right) \right] \right)^{l_{k-1}} \\
\cdot \; & v\left\{ -\frac{\varepsilon_0}{4} \leq \frac{\bar{B}(n_k - n_{k-1} - N_{k-1} l_{k-1})}{\sqrt{2n_k \log\log n_k}} \leq \frac{\varepsilon_0}{4} \right\},
\end{aligned}$$

where $\phi(x)$ is a even function defined by

$$\phi(x) := \begin{cases} 1 - e^{|x| - \varepsilon_0 t/4}, & |x| \leq \varepsilon_0 t/4; \\ 0, & |x| > \varepsilon_0 t/4. \end{cases}$$

Applying Lemma 2.1 and CLT [15], we have if $k \to \infty$

$$\begin{aligned}
\log \; & \mathcal{E}\left[\phi\left(\frac{\bar{B}(N_{k-1})}{r_{k-1}} - (x(\frac{i}{m}) - x(\frac{i-1}{m}))t \right) \right] \to \log \mathcal{E}\left[\phi\left(\bar{B}(1) - (x(\frac{i}{m}) - x(\frac{i-1}{m}))t \right) \right] \\
\geq \; & -\frac{m(x(\frac{i}{m}) - x(\frac{i-1}{m}))^2 t^2}{2\underline{\sigma}^2} + \log \mathcal{E}[\phi(\bar{B}(1))].
\end{aligned}$$

Thus as $k \to \infty$

$$
\begin{aligned}
& \frac{n_k - n_{k-1}}{2n_k \log\log n_k} \cdot \log \mathcal{E}\left[\phi\left(\frac{\bar{B}(N_{k-1})}{r_{k-1}} - (x(\frac{i}{m}) - x(\frac{i-1}{m}))t\right)\right]^{l_{k-1}} \\
= \; & \frac{l_{k-1}(n_k - n_{k-1})}{2n_k \log\log n_k} \cdot \log \mathcal{E}\left[\phi\left(\frac{\bar{B}(N_{k-1})}{r_{k-1}} - (x(\frac{i}{m}) - x(\frac{i-1}{m}))t\right)\right] \\
\to \; & t^{-2} \log \mathcal{E}[\phi(\bar{B}(1) - (x(\frac{i}{m}) - x(\frac{i-1}{m}))t)] \\
\geq \; & -\frac{m(x(\frac{i}{m}) - x(\frac{i-1}{m}))^2}{2\underline{\sigma}^2} + t^{-2} \log \mathcal{E}[\phi(\bar{B}(1))].
\end{aligned}
$$

Together with $\varliminf_{t\to\infty} t^{-2} \log \mathcal{E}[\phi(\bar{B}(1) - (x(\frac{i}{m}) - x(\frac{i-1}{m}))t)] \geq -\frac{m(x(\frac{i}{m})-x(\frac{i-1}{m}))^2}{2\underline{\sigma}^2}$, we have, for large enough t,

$$
\begin{aligned}
\lim_{k\to\infty} \frac{n_k - n_{k-1}}{2n_k \log\log n_k} \cdot \log \mathcal{E}\left[\phi\left(\frac{\bar{B}(N_{k-1})}{r_{k-1}} - (x(\frac{i}{m}) - x(\frac{i-1}{m}))t\right)\right]^{l_{k-1}} \\
\geq -\frac{m(x(\frac{i}{m}) - x(\frac{i-1}{m}))^2}{2\underline{\sigma}^2} - 1. \qquad (6)
\end{aligned}
$$

On the other hand, by Chebyshev's inequality, it follows that

$$
V(|\frac{\bar{B}(n_k - n_{k-1} - N_{k-1}l_{k-1}))}{\sqrt{2n_k \log\log n_k}}| > \frac{\varepsilon_0}{4}) \leq \frac{8(n_k - n_{k-1} - N_{k-1}l_{k-1})\bar{\sigma}^2}{m\varepsilon_0^2 n_k \log\log n_k} \to 0, \; k \to \infty.
$$

Therefore, as $k \to \infty$,

$$
\begin{aligned}
& \frac{n_k - n_{k-1}}{2n_k \log\log n_k} \cdot \log v\left\{-\frac{\varepsilon_0}{4} \leq \frac{\bar{B}(n_k - n_{k-1} - N_{k-1}l_{k-1}))}{\sqrt{2n_k \log\log n_k}} \leq \frac{\varepsilon_0}{4}\right\} \\
= & \frac{n_k - n_{k-1}}{2n_k \log\log n_k} \cdot \log\left(1 - V(|\frac{\bar{B}(n_k - n_{k-1} - N_{k-1}l_{k-1}))}{\sqrt{2n_k \log\log n_k}}| > \frac{\varepsilon_0}{4})\right) \to 0. \qquad (7)
\end{aligned}
$$

So, from (6) and (7) we have

$$
\begin{aligned}
& \varliminf_{k\to\infty} \frac{n_k - n_{k-1}}{2n_k \log\log n_k} \cdot \log v\left\{\left|\frac{\bar{B}(n_k) - \bar{B}(n_{k-1})}{\sqrt{2n_k \log\log n_k}} - (x(\frac{i}{m}) - x(\frac{i-1}{m}))\right| \leq \frac{\varepsilon_0}{2}\right\} \\
\geq \; & -\frac{m(x(\frac{i}{m}) - x(\frac{i-1}{m}))^2}{2\underline{\sigma}^2} - 1.
\end{aligned}
$$

Since $x \in K_\beta$, we have

$$
|x(\frac{i}{m}) - x(\frac{i-1}{m})| \leq \frac{\beta}{\sqrt{m}} < \frac{\underline{\sigma}}{\sqrt{m}}, \quad \forall i = 1, \cdots, m.
$$

So there exist $\delta > 0$ such that $d := \delta + \max_{i\le m}(\frac{x(\frac{i}{m})-x(\frac{i-1}{m})}{\frac{\sigma}{\sqrt{m}}})^2 + 1 < 2$. Thus, there exist k_0 such that $\forall k \ge k_0$,

$$\begin{aligned}
&v\left\{\left|\frac{\bar{B}(n_k)-\bar{B}(n_{k-1})}{\sqrt{2n_k\log\log n_k}} - (x(\frac{i}{m}) - x(\frac{i-1}{m}))\right| \le \frac{\varepsilon_0}{2}\right\}\\
&\ge \exp(-dn_k\log\log n_k/(n_k-n_{k-1})).
\end{aligned}$$

Meanwhile from the LIL of chen [2] we have $v(\overline{\lim}_{n\to\infty}\frac{\bar{B}(n)}{\sqrt{2n\log\log n}} \le \frac{\bar{\sigma}}{\sqrt{m}}) = 1$ and $\frac{\sqrt{2n_{k-1}\log\log n_{k-1}}}{\sqrt{2n_k\log\log n_k}} \to 0$ as $k\to\infty$, hence there exist k_1 such that $\forall k \ge k_1$,

$$v\left\{\frac{|\bar{B}(n_{k-1})|}{\sqrt{2n_{k-1}\log\log n_{k-1}}}\frac{\sqrt{2n_{k-1}\log\log n_{k-1}}}{\sqrt{2n_k\log\log n_k}} < \frac{\varepsilon_0}{2}\right\} \ge \frac{1}{2}.$$

By applying lemma 2.3, we get for any $k > (k_0 \vee k_1)$

$$\begin{aligned}
v(A_{n_k}) &= \prod_{i=1}^{m} v(A_{n_k}^i)\\
&\ge \frac{1}{2^m}\exp(-mdn_k\log\log n_k/(n_k-n_{k-1}))\\
&\ge \frac{1}{2^m}\exp(-2mn_k\log\log n_k/(n_k-n_{k-1}))\\
&\sim \frac{1}{2^m}\exp(-2m\log\log n_k)\\
&\sim \frac{1}{k^{2m\alpha}(2\log k)^{2m}}.
\end{aligned}$$

Thus $\sum_{k=1}^{\infty} v(A_{n_k}) = \infty$ for $2m\alpha < 1$, using the Borel-Cantelli lemma, we get infinitely many events A_{n_k} happen q.s..

Next will show that for any given $0 < s < t < 1$, quasi-surely there is infinitely k such that

$$\zeta_{n_k}(t) - \zeta_{n_k}(s) \le \bar{\sigma}\sqrt{t-s} + \varepsilon_0. \tag{8}$$

In fact, by the LIL under capacity, we have

$$v(\overline{\lim_{n\to\infty}}\frac{|\tilde{B}(n)|}{\sqrt{2n\log\log n}} \le \bar{\sigma}\sqrt{t-s}) = 1,$$

where $\tilde{B}(r) = B((t-s)r) \sim \mathcal{N}(0,[(t-s)\underline{\sigma}^2 r,(t-s)\bar{\sigma}^2 r])$. So, we have

$$\begin{aligned}
&v\{\bigcap_{i=1}^{\infty}\bigcup_{k=i}^{\infty}(|\zeta_{n_k}(t)-\zeta_{n_k}(s)| \le \bar{\sigma}\sqrt{t-s}+\varepsilon_0)\}\\
&= v\{\bigcap_{i=1}^{\infty}\bigcup_{k=i}^{\infty}(\frac{|\tilde{B}(n_k)|}{\sqrt{2n_k\log\log n_k}} \le \bar{\sigma}\sqrt{t-s}+\varepsilon_0)\} = 1.
\end{aligned}$$

For any $\varepsilon > 0$ we set $m > (\frac{4\bar{\sigma}}{\varepsilon})^2$ and choose ε_0 such that $\varepsilon_0 < \frac{\varepsilon}{2m}$ after fixed m. Now we consider the n_k making A_{n_k} happens and satisfying inequality (8),

$$
\begin{aligned}
&\|\zeta_{n_k} - x\| = \sup_{t\in[0,1]} |\zeta_{n_k}(t) - x(t)| \\
&= \sup_{t\in[0,1]} \Bigg| \zeta_{n_k}(t) - \zeta_{n_k}(\frac{\lfloor mt \rfloor}{m}) + x(\frac{\lfloor mt \rfloor}{m}) - x(t) \\
&+ \sum_{i=1}^{\lfloor mt \rfloor} \left[\zeta_{n_k}(\frac{i}{m}) - \zeta_{n_k}(\frac{i-1}{m}) - \left(x(\frac{i}{m}) - x(\frac{i-1}{m}) \right) \right] \Bigg| \\
&\le \frac{\bar{\sigma}}{\sqrt{m}} + \frac{\beta}{\sqrt{m}} + m\varepsilon_0 \\
&\le \frac{2\bar{\sigma}}{\sqrt{m}} + m\varepsilon_0 \\
&\le \varepsilon.
\end{aligned}
$$

We conclude that

$$v\{x \in C(\zeta_n)\} = 1,$$

and thus

$$v\{K_\beta \subseteq C(\zeta_n)\} = 1,$$

for K_β having countable dense set. So

$$v\{\bigcup_{n=1}^{\infty} K_{\underline{\sigma}-\frac{1}{n}} \subseteq C(\zeta_n)\} = 1.$$

Since $C(\zeta_n))$ is a closed set, we get

$$v\{K_{\underline{\sigma}} \subseteq C(\zeta_n)\} = 1.$$

The proof of (III) is complete.

(IV). For any $\beta \in [\underline{\sigma}, \bar{\sigma}]$, there exist $P_\beta \in \mathcal{P}$ such that $B(t)/\beta$ be a classical Brownian motion under P_β, so by the stranssen's invariance principle $P_\beta(C(B_n) = K_\beta) = 1$. Therefore, $V(C(B_n) = K_\beta) = 1$.

The proof of Theorem 3.1 is complete. $\square$

Notice that the discreteness of n is inessential for the previous considerations. More precisely, the following corollary holds true.

Corollary 3.1 *If $u > e$ is real and we put $\zeta_u(t) = (2u \log\log u)^{-\frac{1}{2}} B(ut),\ t \in [0,1]$, then we have*
(I) The sequence $(\zeta_u)_{u>e}$ is relatively norm-compact q.s..
(II) $v\{C(\zeta_u) \subseteq K_{\bar{\sigma}}\} = 1$.
(III) $v\{C(\zeta_u) \supseteq K_{\underline{\sigma}}\} = 1$.
(IV) $\forall \beta \in [\underline{\sigma}, \bar{\sigma}], V\{C(\zeta_u) = K_\beta\} = 1$.

Corollary 3.2 *If φ is a continuous map from $C[0,1]$ to some Hausdorff space H, then we have*
(I) The sequence $(\varphi(\zeta_n))_{n\geq 3}$ is relatively norm-compact q.s..
(II) $v\{C(\varphi(\zeta_n)) \subseteq \varphi(K_{\bar{\sigma}})\} = 1$.
(III) $v\{C(\varphi(\zeta_n)) \supseteq \varphi(K_{\underline{\sigma}})\} = 1$.
(IV) $\forall \beta \in [\underline{\sigma}, \bar{\sigma}], V\{C(\varphi(\zeta_n)) = \varphi(K_\beta)\} = 1$.
We substitute $(\zeta_n)_{n\geq 3}$ with $(\zeta_u)_{u>e}$, the conclusion also holds.

4 Some applications and comments

In this section, we give some applications and comments, which can be obtained by our invariance principle and the arguments of Strassen [17].

Example 4.1 *Let $f(\cdot)$ be any Riemann integrable real function on $[0,1]$,*

$$F(t) = \int_t^1 f(s)ds,\ t \in [0,1].$$

Then,

$$v\left\{\underline{\sigma}\left(\int_0^1 F^2(t)dt\right)^{1/2} \leq \overline{\lim_{n\to\infty}}(2n^3\log\log n)^{-1/2}\sum_{i=1}^n f(\frac{i}{n})B_i \leq \bar{\sigma}\left(\int_0^1 F^2(t)dt\right)^{1/2}\right\} = 1.$$

In particular, for any $\alpha > -1$, putting $f(t) = t^\alpha$, we have

$$v\left\{\frac{\underline{\sigma}}{\sqrt{(\alpha+3/2)(\alpha+2)}} \leq \overline{\lim_{n\to\infty}}(2n^{2\alpha+3}\log\log n)^{-1/2}\sum_{i=1}^n i^\alpha B_i \leq \frac{\bar{\sigma}}{\sqrt{(\alpha+3/2)(\alpha+2)}}\right\} = 1.$$

Example 4.2 *Let $a \geq 1$ be real, then we have*

$$v\left\{\frac{2(a+2)^{(a/2)-1}}{a^{a/2}\left(\int_0^1 \frac{dt}{\underline{\sigma}\sqrt{1-t^a}}\right)^a} \leq \overline{\lim_{n\to\infty}} n^{-1-(a/2)}(2\log\log n)^{-1/2}\sum_{i=1}^n |B_i|^a \leq \frac{2(a+2)^{(a/2)-1}}{a^{a/2}\left(\int_0^1 \frac{dt}{\bar{\sigma}\sqrt{1-t^a}}\right)^a}\right\} = 1.$$

In particular, a=1,2, we have

$$v\left\{\frac{\underline{\sigma}}{\sqrt{3}} \leq \overline{\lim_{n\to\infty}} n^{-3/2}(2\log\log n)^{-1/2}\sum_{i=1}^n |B_i| \leq \frac{\bar{\sigma}}{\sqrt{3}}\right\} = 1,$$

$$v\left\{\frac{4\underline{\sigma}^2}{\pi^2} \leq \overline{\lim_{n\to\infty}} n^{-2}(2\log\log n)^{-1}\sum_{i=1}^n |B_i|^2 \leq \frac{4\bar{\sigma}^2}{\pi^2}\right\} = 1.$$

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